Mathematics and Problem Solving

Lecture 8.2

Proof by Cases

Proof by Cases

- Sometimes proofs can be broken down into **cases**
- It can be easier to prove the cases one by one
- If all the cases are proved, then the claim is proved

- Cases must be exhaustive
  - Number is odd or even
  - Number is rational or irrational
  - x >= 0 or x < 0
  - …
- All of them must be proved

- **Claim:** None of the shoes shop X sells fit
- **“Proof”:** Shop X sells sandals, kitten heels and brogues.
  - Case 1: The sandals are too loose, thus they do not fit
  - Case 2: The kitten heels are too narrow in the toe box, thus they do not fit
  - Case 3: The brogues chafe in the heel, thus they do not fit
  - All of the types of shoe sold by shop X do not fit, therefore none of the shoes sold by shop X fit. ∎

Proof by Exhaustion

- **Proof by Exhaustion** – Show true for every possible case
- **Theorem:** $x + 3 < 10$ for integers $0 < x < 4$
- **Proof:**
  - Assume $x=1$, observe that $1+3 = 4 < 10$
  - Assume $x=2$, observe that $2+3 = 5 < 10$
  - Assume $x=3$, observe that $3+3 = 6 < 10$
  - Thus $x + 3 < 10$ for all integers $0 < x < 4$ ∎

Proof by Cases in Propositional Logic

- $P \vee Q \rightarrow X$
  - Assume P, show X
    - (i.e. prove $P \rightarrow X$)
  - Assume Q, show X
    - (i.e. prove $Q \rightarrow X$)

Example

- **Theorem:** 3(2x + 2) + 1 is odd for all x
- **Proof:**
  - Case 1: Assume x is even, therefore x = 2k
    - 3(2(2k)+2)+1 = 3(4k + 2)+1 = 12k + 7 = 2(6k + 3) + 1
    - 2(6k + 3) + 1 is odd
    - Therefore if x is even, 3(2(2k)+2)+1 is odd
  - Case 2: Assume x is odd, therefore x = 2k+1
    - 3(2(2k +1)+2)+1 = 3(4k + 4)+1 = 12k + 13 = 2(6k + 6) + 1
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Summary

- Proof by Cases
  - Divide into cases and prove for each
- Proof by Exhaustion
  - Show it is true for every possible value