Mathematics and Problem Solving

Lecture 8.3

Proof by Contradiction

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- Assume our conjecture is false and show this leads to contradiction

Example

- **Claim:** The tallest person in the world is taller than me (and I am not the tallest person)
- **“Proof”:** Assume the tallest person in the world is *not* taller than me
  - Thus I would be taller than the tallest person.
  - Thus I would be both the tallest person and not the tallest person
  - Therefore by assuming the tallest person in the world is not taller than me, we conclude by contradiction that the tallest person is taller than me. ∎

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Contradiction in Propositional Logic

- Contradiction, e.g. $P \wedge \neg P$ is equivalent to false
- If $P \rightarrow$ false
- Then P must be false

Example

- **Theorem:** sqrt(2) is irrational
- **Proof:** Assume sqrt(2) is not irrational, therefore sqrt(2) is rational. Thus sqrt(2) = a/b (where a and b are in lowest terms)
  - $2 = a^2 / b^2$
  - $2b^2 = a^2$; thus a^2 is even, therefore a is even, so a=2k
  - $2b^2 = (2k)^2 = 4k^2$
  - $b^2 = 2k^2$; thus b^2 is even, therefore b is even,
  - If a and b are even a/b is not a fraction in its lowest terms (this contradicts our earlier assumption)
  - By assuming sqrt(2) is rational, we conclude by contradiction that sqrt(2) is not rational, thus root(2) is irrational ∎

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