Mathematics and Problem Solving

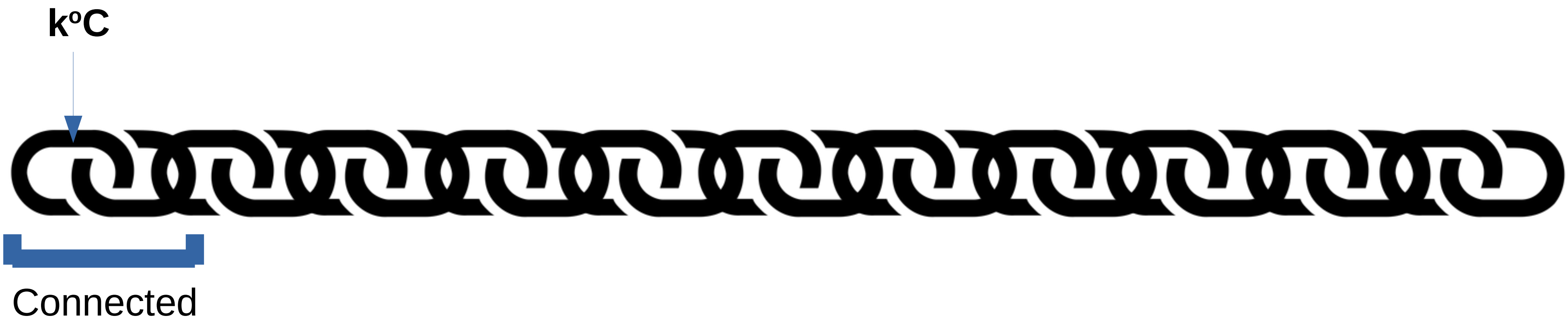
Lecture 8.4

Proof by Induction

- Proof by induction
  - Prove something for an (infinite) set by
    1) proving it for one element, and
    2) proving a rule that states “if it is true for the kth element, it is true for the k+1th element”

Chain example

- **Claim:** Given enough time a chain will become a constant temperature if there is a link that doesn't change temperature.
- **"Proof":** One link in the chain remains a constant temperature. Given enough time, any connected pair of chain links will become the same temperature. Therefore, the whole chain will become a constant temperature. ∎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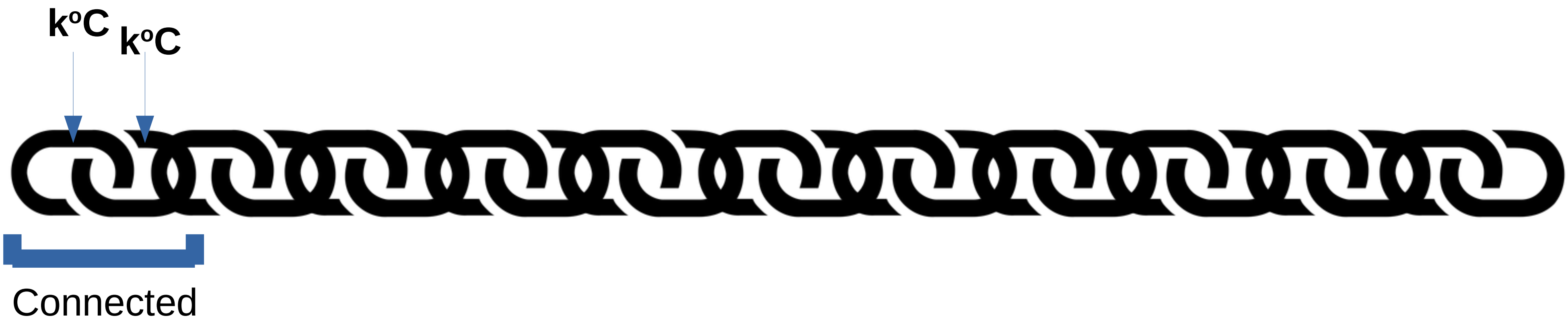
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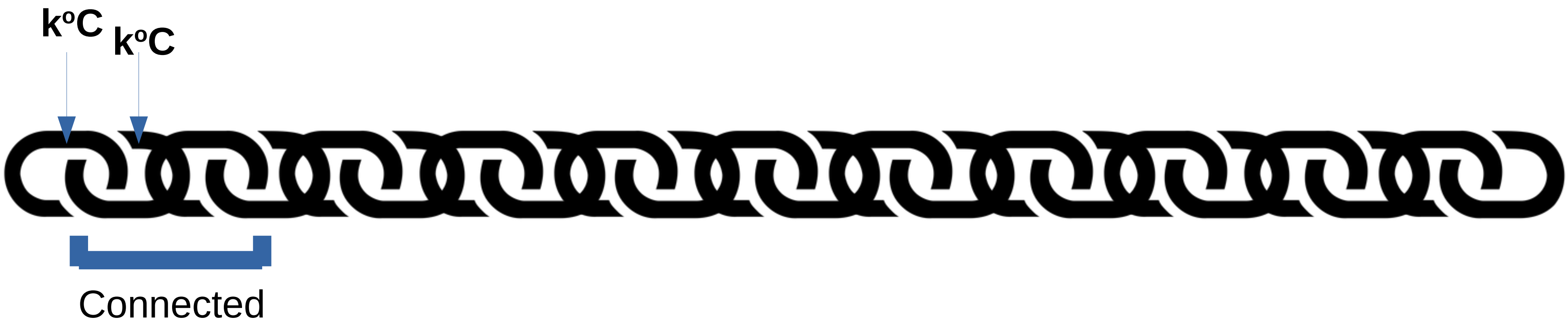

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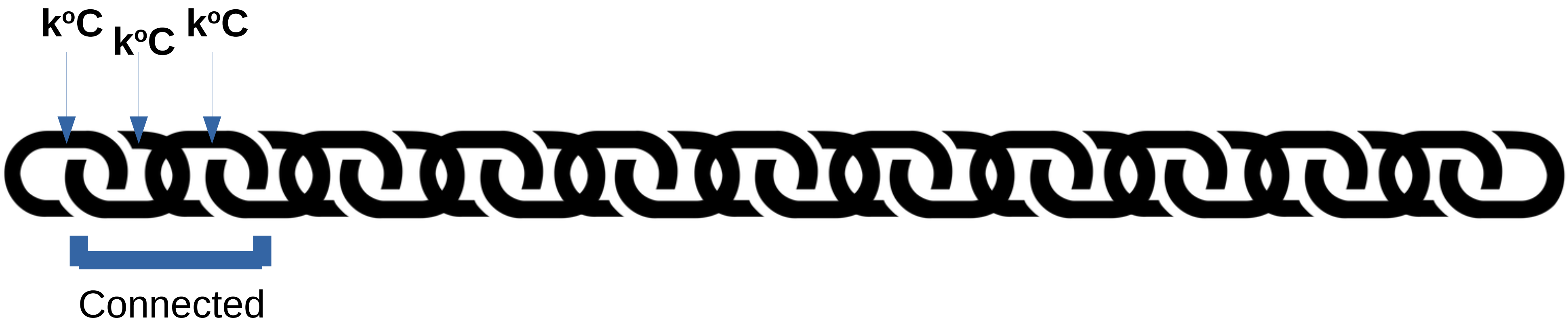

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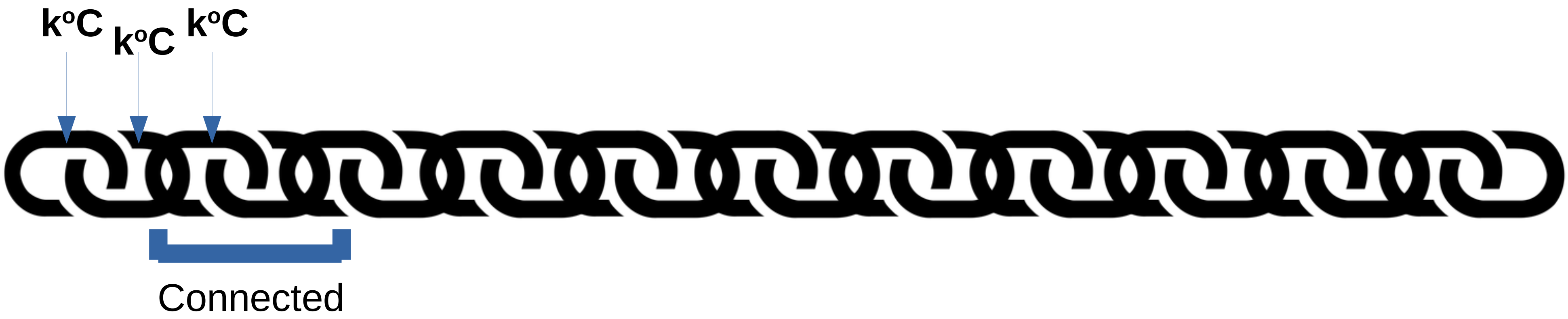
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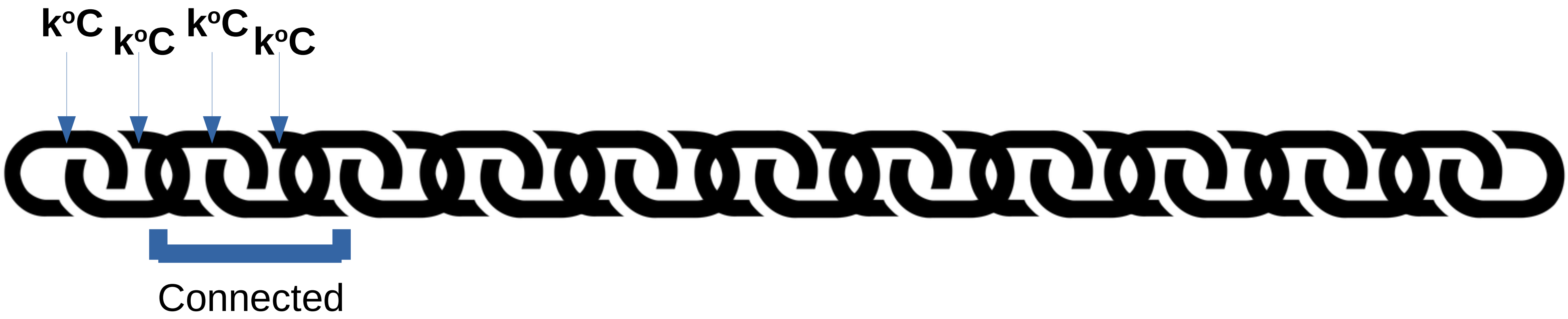

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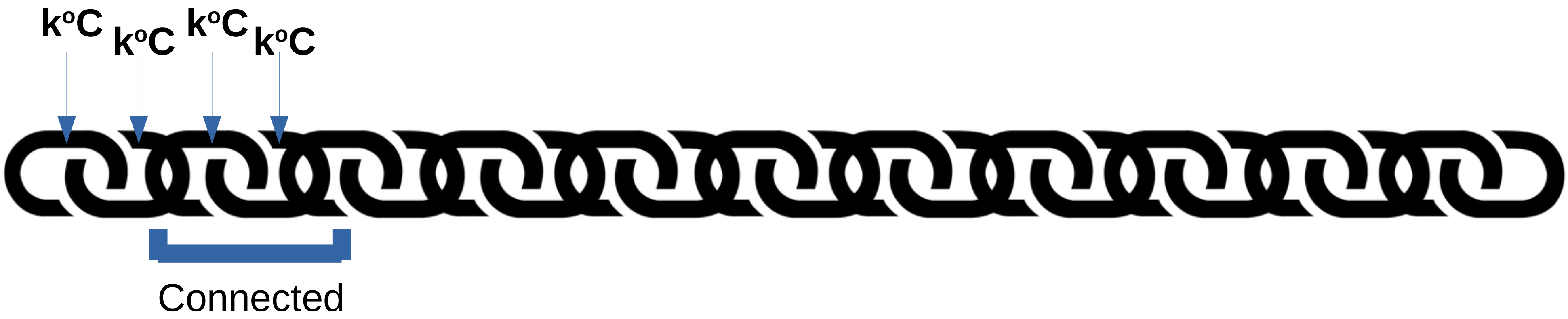

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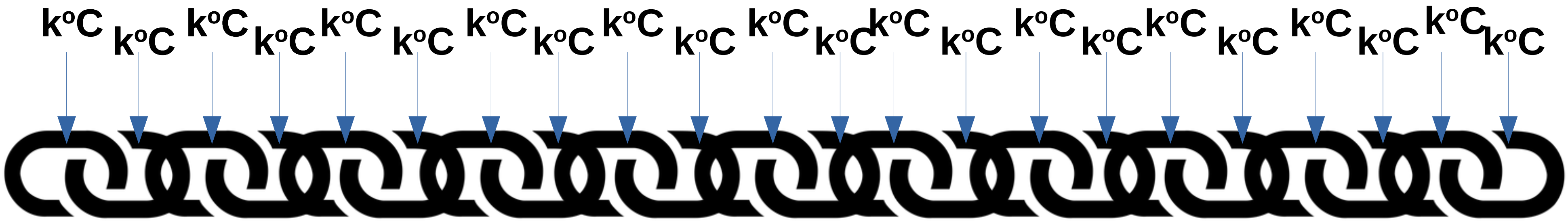
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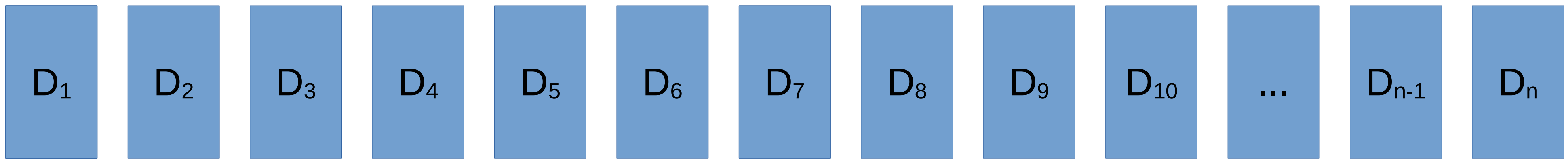
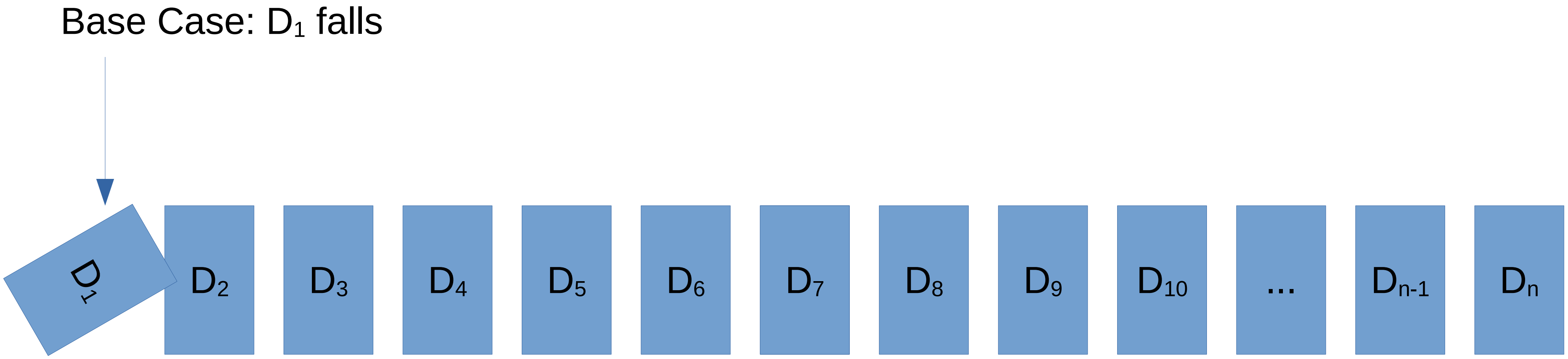
- **Base Case:** One link in the chain remains a constant temperature.
  - Claim is true for one element
- **Induction Step:** Connected links become the same temperature
  - If claim is true for an element k_1 it is also true for another element k_2

Domino Example

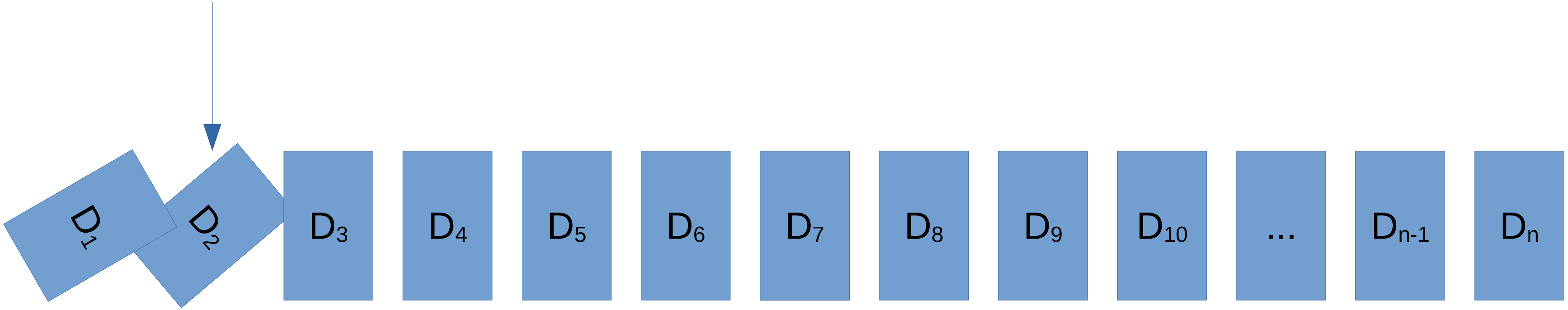
- **Claim:** If I knock over the first domino in a run, all of the dominoes fall over
- **“Proof”:** As I knock over the first domino in the run, it will fall. If a domino falls, the next domino will also fall. Therefore, all the dominoes will fall. ∎



D_1
D_2
D_3
D_4
D_5
D_6
D_7
D_8
D_9
D_{10}
...
D_{n-1}
D_n


Base Case: D_1 falls
D_1
D_2
D_3
D_4
D_5
D_6
D_7
D_8
D_9
D_{10}
...
D_{n-1}
D_n

Induction Step: If D_k falls, D_{k+1} falls



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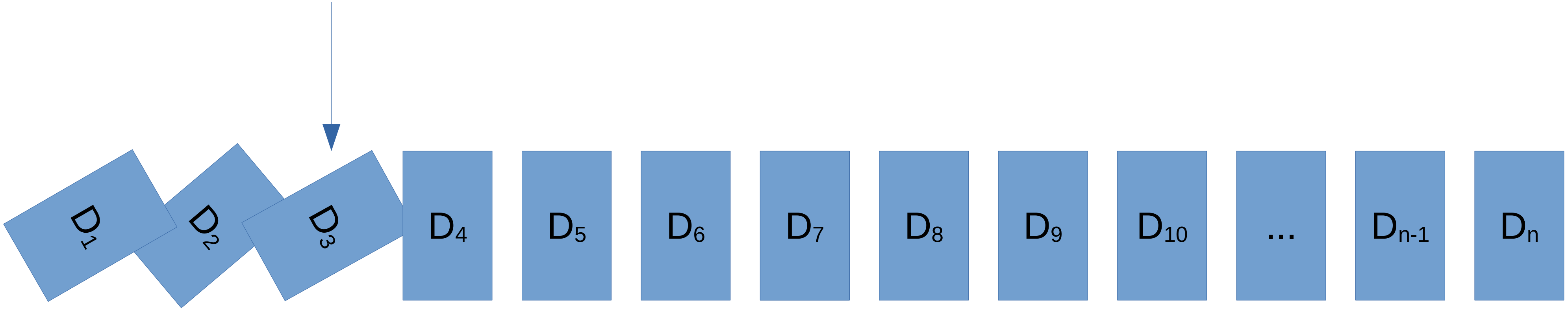
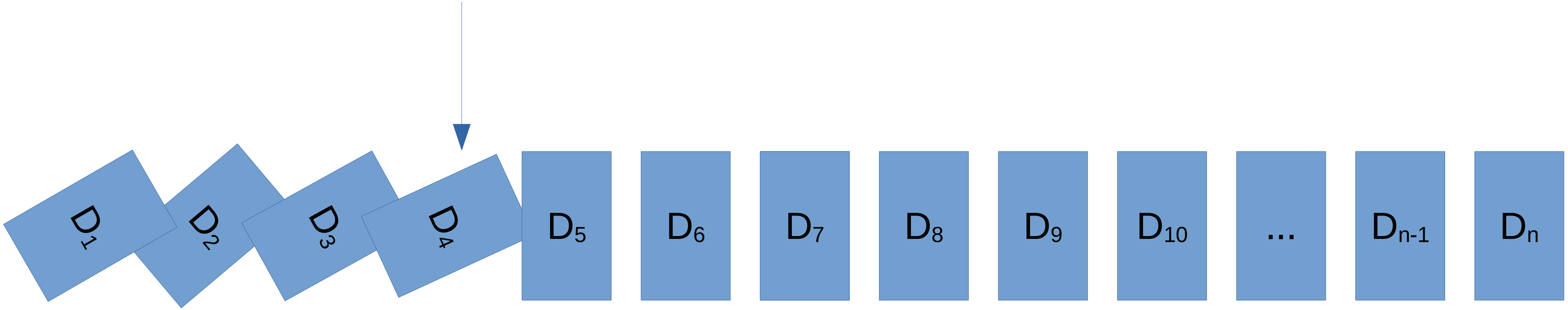
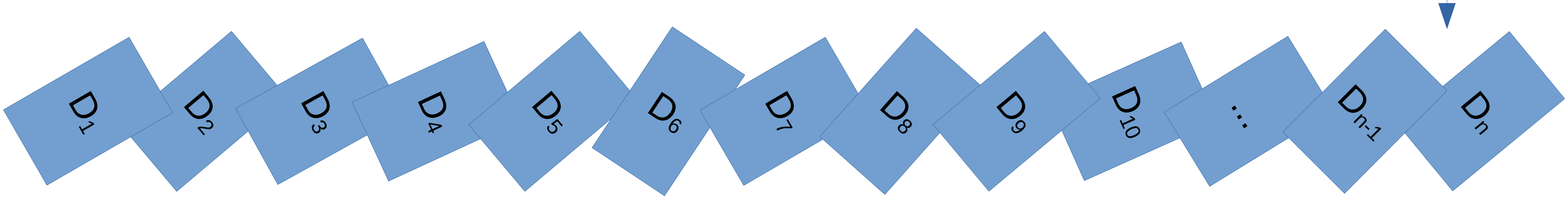


Induction Step: If D_k falls, D_{k+1} falls



Induction Step: If D_{n-1} falls, D_n falls



- **Base Case:** First domino falls
  - Claim is true for one element
- **Induction Step:** If one domino falls, the next will fall
  - If claim is true for an element k_1 it is also true for another element k_2

Sum of n natural numbers

- **Theorem:** $S(n) = 1 + 2 + 3 + \ldots + (n-1) + n = n(n+1)/2$
- **Proof:** We will prove a base case where n=1 and an show that if the theorem is correct for S(k) then it is correct for S(k+1).
  - Base case: Assume n is 1, then $1(1+1)/2 = 2/2 = 1$, which is the sum of all numbers up to 1.
  - Inductive step: Assume the theorem is true for n = k. The sum of the first k+1 numbers will be given by the formula:
    - $S(k) + k+1$
    - $= k(k+1)/2 + k + 1$
    - $= k(k+1)/2 + 2(k + 1)/2$
    - $= (k(k+1) + 2(k + 1))/2$
    - $= ((k+1)(k+2))/2$
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